The Hamiltonian theory of the Landau–Lifschitz equation with an easy axis

2004 J. Phys. A: Math. Gen. 37 6311

(http://iopscience.iop.org/0305-4470/37/24/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 6311–6318 PII: S0305-4470(04)71685-6

The Hamiltonian theory of the Landau–Lifschitz equation with an easy axis

Jinchun He[1], Lina Shi[2], Hua Chen[1] and Nianning Huang[2]

[1] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, People's Republic of China

[2] School of Physics, Wuhan University, Wuhan 430072, People's Republic of China

E-mail: he_jc@sohu.com and shilinashilina@163.com

Received 9 November 2003
Published 2 June 2004
Online at stacks.iop.org/JPhysA/37/6311
doi:10.1088/0305-4470/37/24/009

Abstract

We obtain the Hamiltonian theory of the Landau–Lifschitz equation with an easy axis by using a suitable gauge transformation and the standard procedure. Action–angle variables are found and the canonical equation is given.

PACS numbers: 05.45.Yv, 02.30.Zz, 75.10.Pq

1. Introduction

The equation of a continuous one-dimensional ferromagnet, known as the Landau–Lifschitz equation (L–L equation for short) [1–3], has attracted much attention in the past decades [4–7]. Formulation of the Hamiltonian formalism of the equation for the isotropic case was given, but some problems remain open [8].

The exact solutions [9] of the equation for the spin chain with an easy axis were obtained in 1995 [7] by using the inverse scattering method. Hence it is time now to formulate its Hamiltonian formalism [10]. Usually, one would follow the following method: after introducing the Lie–Poisson bracket for the components of a spin variable [1], the Hamiltonian equation is established and the Hamiltonian can be uniquely represented in the form of an integral with respect to x [8]. Next, following the standard procedure, the Lie–Poisson bracket for the two components of the monodromy matrix are derived. The action–angle variables are then constructed and their Lie–Poisson bracket is determined. Hence the Hamiltonian function in the form of an integral with respect to the spectral parameter is expressed by taking into account the time dependence of the angle variable.

The next problem is to find a conserved quantity [11] which has two different integral representations where the integrations are in x and in the spectral parameter respectively, and these two integral representations should concide with the two integral forms of the Hamiltonian.

The same problem also appears in the case of an isotropic spin chain: it was finally solved after the equation for isotropic spin chain was shown to be gauge equivalent to the nonlinear Schrödinger equation [12]. Therefore, Hamiltonians in the form of integrals in the spectral parameter for these two equations are the same and those in the form of an integral in x are equivalent on account of gauge equivalence.

In this paper, we will apply this procedure to the L–L equation. However, there does not exist any equation which is gauge equivalent to the L–L equation with an easy axis playing the same role as the NLS equation in the case of the L–L equation of the isotropic spin chain. To solve the problem in the present case a gauge transformation is chosen to rotate the spin chain with arbitrary direction in spin space to the third axis; the conserved quantities are then shown to have the desired form of integral in x of the Hamiltonian. Finally, when the discrete spectrum of the spectral parameter is added, the Hamiltonian formalism is obtained.

The L–L equation is

$$\frac{\partial \overrightarrow{S}}{\partial t} = \overrightarrow{S} \wedge \frac{\partial^2 \overrightarrow{S}}{\partial x^2} + \overrightarrow{S} \wedge J \overrightarrow{S} \tag{1}$$

where $J = \mathrm{diag}(J_1, J_2, J_3)$, $J_1 \leqslant J_2 \leqslant J_3$, and $\overrightarrow{S}$ is the spin which is a vector with $\overrightarrow{S} \cdot \overrightarrow{S} = 1$. In the case of an easy axis, i.e. $0 = J_1 = J_2 < J_3 = J$.

For the spin chain, we introduce the Lie–Poisson bracket

$$\{S_\alpha(x), S_\beta(y)\} = -\epsilon_{\alpha\beta\gamma} S_\gamma(x)\delta(x-y) \tag{2}$$

where $\epsilon_{\alpha\beta\gamma}$ is the totally skew-symmetric rank 3 tensor. $\alpha, \beta, \gamma = 1, 2, 3$, and having two equal indices means we are summing over this index. Using the Lie–Poisson bracket, the Hamiltonian equation of (1) can be written as

$$\frac{\partial \overrightarrow{S}}{\partial t} = \{H, \overrightarrow{S}\} \tag{3}$$

where the Hamiltonian H is determined uniquely as

$$H = \int_{-\infty}^{+\infty} \mathrm{d}x \big(\overrightarrow{S_x} \cdot \overrightarrow{S_x} - J_3 S_3^2 \big). \tag{4}$$

The compatibility pairs of the L–L equation are

$$L = -\mathrm{i} u_\alpha S_\alpha \sigma_\alpha \qquad M = 2\mathrm{i} \frac{u_1 u_2 u_3}{u_\alpha} S_\alpha \sigma_\alpha - \mathrm{i} u_\alpha \epsilon_{\alpha\beta\gamma} S_\beta \frac{\partial S_\gamma}{\partial x} \sigma_\alpha \tag{5}$$

where u_α is the characteristic parameter of the spin chain. In the case of an easy axis, $u_1 = u_2 = \lambda$, $u_3 = \kappa$, $\lambda^2 - \kappa^2 = \rho^2$. We can introduce an affine parameter ζ

$$\lambda = \tfrac{1}{2}(\zeta + \rho^2 \zeta^{-1}) \qquad \kappa = \tfrac{1}{2}(\zeta - \rho^2 \zeta^{-1}) \tag{6}$$

to account for the multivaluedness of λ and κ.

2. The Lie–Poisson brackets of the elements of the monodromy matrix

The tensor product of two monodromy matrices introduced by Zhakharov is $\{T(\zeta) \otimes T^{-1}(\zeta')\}_{il,jm} = T(\zeta)_{ij} \otimes T^{-1}(\zeta')_{lm}$, hence,

$$\{T(\zeta) \otimes T^{-1}(\zeta')\} = -\epsilon_{\alpha\beta\gamma} \int \mathrm{d}x \frac{\delta T(\zeta)}{\delta S_\alpha(x)} \otimes \frac{\delta T^{-1}(\zeta')}{\delta S_\beta(x)} S_\gamma(x) \tag{7}$$

where $T(\zeta)$ is the monodromy matrix. The right-hand side of (7) can be expressed as Jost functions, i.e.

$$\{T(\zeta) \otimes T^{-1}(\zeta')\} = \int \mathrm{d}x \Psi^{-1}(x,\zeta)\Phi^{-1}(x,\zeta')R\Psi(x,\zeta')\Phi(x,\zeta) \tag{8}$$

where

$$\begin{aligned} R = {} & S_3(u_1u_2'\sigma_1 \otimes \sigma_2 - u_2u_1'\sigma_2 \otimes \sigma_1) + S_1(u_2u_3'\sigma_2 \otimes \sigma_3 - u_3u_2'\sigma_3 \otimes \sigma_2) \\ & + S_2(u_3u_1'\sigma_3 \otimes \sigma_1 - u_1u_3'\sigma_1 \otimes \sigma_3). \end{aligned}$$

The integral (8) can be written in closed form if the integrand is an exact derivative. We try to write the derivative as

$$\partial_x(f_0\Psi^{-1}(\zeta)\Psi(\zeta') \otimes \Phi^{-1}(\zeta')\Phi(\zeta) + f_3\Psi^{-1}(\zeta)\sigma_3\Psi(\zeta') \otimes \Phi^{-1}(\zeta')\sigma_3\Phi(\zeta)) \tag{9}$$

where f_0 and f_3 are to be determined. From the first equation of (5), equation (9) becomes

$$\Psi^{-1}(x,\zeta)\Phi^{-1}(x,\zeta')W\Psi(x,\zeta')\Phi(x,\zeta) \tag{10}$$

where $W = W_0 + W_3$, and W_0 and W_3 are respectively

$$W_0 = f_0\mathrm{i}(u_\alpha - u_\alpha')S_\alpha(\sigma_\alpha \otimes' I - I \otimes' \sigma_\alpha)$$

$$\begin{aligned} W_3 = {} & f_3[\mathrm{i}S_1(u_1 + u_1')(\mathrm{i}\sigma_2 \otimes' \sigma_3 + \sigma_3 \otimes' \mathrm{i}\sigma_2) + \mathrm{i}S_2(u_2 + u_2')(\mathrm{i}\sigma_1 \otimes' \sigma_3 + \sigma_3 \otimes' \mathrm{i}\sigma_1) \\ & - \mathrm{i}S_3(u_3 - u_3')(\sigma_3 \otimes' I - I \otimes' \sigma_3)]. \end{aligned}$$

In the above equations $\otimes'$ is another tensor product defined as $A_{im}B_{lj} = (A \otimes' B)_{il,jm}$. We can determine the coefficients f_0, f_3 by comparing the integrand in (8) with (10). Therefore

$$f_0 = -\frac{1}{2}\frac{\zeta\zeta' + \rho^2}{\zeta - \zeta'} \qquad f_3 = \frac{1}{2}\rho^2\frac{(\zeta - \zeta')\zeta^{-1}\zeta'^{-1}}{1 + \rho^2\zeta^{-1}\zeta'^{-1}}. \tag{11}$$

After a lengthy calculation we find that the right-hand side of (8) is the sum of the following two formulae:

$$f_0\begin{pmatrix} 0 & -a\tilde{b}' & -\tilde{a}'\tilde{b} & 0 \\ -b'a & 0 & 0 & \tilde{b}a' \\ -b\tilde{a}' & 0 & 0 & \tilde{b}'\tilde{a} \\ 0 & a'b & \tilde{a}b' & 0 \end{pmatrix} - \frac{1}{2}(\zeta'\zeta + \rho^2)\mathrm{i}\pi\delta(\zeta - \zeta')\begin{pmatrix} 0 & \tilde{b}'a' & \tilde{b}'\tilde{a} & 0 \\ a'b & 0 & 2|a|^2 & a\tilde{b}' \\ \tilde{a}b' & -2|a|^2 & 0 & \tilde{a}'\tilde{b} \\ 0 & b'a & \tilde{a}'b & 0 \end{pmatrix} \tag{12}$$

and

$$\begin{aligned} & f_3\begin{pmatrix} 0 & -a\tilde{b}' & -\tilde{a}'\tilde{b} & 0 \\ -b'a & 0 & 0 & \tilde{b}a' \\ -b\tilde{a}' & 0 & 0 & \tilde{b}'\tilde{a} \\ 0 & a'b & \tilde{a}b' & 0 \end{pmatrix} \\ & \qquad - \frac{1}{2}\rho^2(\zeta - \zeta')\zeta'^{-1}\delta(\zeta + \rho^2\zeta'^{-1})\begin{pmatrix} 0 & \tilde{b}'a' & \tilde{b}'\tilde{a} & 0 \\ a'b & 0 & 2|a|^2 & a\tilde{b}' \\ \tilde{a}b' & -2|a|^2 & 0 & \tilde{a}'\tilde{b} \\ 0 & b'a & \tilde{a}'b & 0 \end{pmatrix}. \end{aligned} \tag{13}$$

Therefore, from (12) and (13), we obtain the Lie–Poisson bracket between any elements of the monodromy matrix, as

$$\begin{aligned} \{a, b'\} = {} & \left[-\frac{1}{2}(\zeta'\zeta + \rho^2)\left(\frac{1}{\zeta - \zeta'} - \mathrm{i}\pi\delta(\zeta - \zeta')\right)\right. \\ & \left. + \frac{1}{2}\rho^2(\zeta - \zeta')\zeta'^{-1}\left(\frac{1}{\zeta + \rho^2\zeta'^{-1}} + \mathrm{i}\pi\delta(\zeta + \rho^2\zeta'^{-1})\right)\right]ab'. \end{aligned} \tag{14}$$

3. The action and angle variables of the continuous spectrum

We will discuss the Hamiltonian which contains the continuous spectral parameter. From the inverse scattering method, we know that $a(\zeta)$, $\tilde{a}(\zeta)$ are independent of the time variable t, but $b(\zeta)$, $\tilde{b}(\zeta)$ depend on t. Since $M \to M_0 = \mathrm{i}\lambda^2\sigma_3$ as $|x| \to \infty$, the inverse scattering method gives $b(t, \zeta) = b(0, \zeta)\,\mathrm{e}^{\mathrm{i}4\lambda^2 t}$.

Therefore, the action variable $P(\zeta)$ must be a function of $a(\zeta)$ and $\tilde{a}(\zeta)$. Suppose that the action variable is $P(\zeta) = F(|a(\zeta)|^2)$, where F is a function to be determined, and the angle variable is $Q(\zeta) = \arg b(\zeta) = \frac{1}{2\mathrm{i}}\ln\frac{b(\zeta)}{\tilde{b}(\zeta)}$. We will determine the form of the function F such that $\{P(\zeta), Q(\zeta')\} = -\delta(\zeta - \zeta')$. Firstly, we point out that through a reduction transformation $\zeta \to -\rho^2\zeta$, we have $L(\zeta) = \sigma_3 L(-\rho^2\zeta)\sigma_3$, $a(\zeta) = a(-\rho^2\zeta)$. Thus, we discuss the case $|\zeta| > \rho$ only because the reduction transformation maps $|\zeta| > \rho$ to $|\zeta| < \rho$. Secondly, from (14) we find $\{|a|^2, b'\} = \mathrm{i}\pi(\zeta^2 + \rho^2)\delta(\zeta - \zeta')|a|^2 b'$. Since

$$\{P(\zeta), Q(\zeta')\} = F'(|a(\zeta)|^2)\pi(\zeta^2 + \rho^2)\delta(\zeta - \zeta')|a(\zeta)|^2 \tag{15}$$

we obtain

$$F'(|a(\zeta)|^2)\pi(\zeta^2 + \rho^2)|a(\zeta)|^2 = 1 \tag{16}$$

hence

$$P(\zeta) = F(|a(\zeta)|^2) = \frac{1}{\pi(\zeta^2 + \rho^2)}\ln|a(\zeta)|^2. \tag{17}$$

From the Hamiltonian equation of $Q(\zeta)$, we obtain $Q_t(\zeta) = \{Q(\zeta, H)\} = 4\lambda^2$. Let H_c be the Hamiltonian which contains the continuous spectral parameter. H_c is unique and can be written as

$$H_c = \int_{|\zeta|>\rho} \mathrm{d}\zeta(4\lambda^2)P(\zeta). \tag{18}$$

We then apply the reduction transformation and see that the Hamiltonian of the L–L equation with an easy axis which contains the continuous spectral parameter is given by

$$H_c = \frac{1}{\pi}\int_{-\infty}^{+\infty} \ln|a(\zeta)|^2\,\mathrm{d}\zeta. \tag{19}$$

4. Conserved quantities

We know that the form of $a(\zeta)$ is independent of the gauge transformation, but the form of a Jost solution does depend on the gauge transformation. So, when deriving $a(\zeta)$ from a Jost solution, we should also choose a suitable gauge transformation such that the limiting behaviour of this Jost solution by taking $x \to \pm\infty$ and then taking the spectral parameter $|\zeta| \to \infty$ should give the same answer as taking $|\zeta| \to \infty$. Therefore, when we succeed in introducing a gauge transformation to transform the first operator of a compatibility pair into the asymptotic form $-\mathrm{i}\frac{1}{2}\zeta\sigma_3 + O(1)$, the conserved quantities are derived, the zeroth one vanishes and the first one has the desired form of Hamiltonian.

Generally, we start from the first operator of the compatibility pair $L = -\mathrm{i}u_\alpha S_\alpha\sigma_\alpha$. We then introduce a gauge transformation $A(x, t)$. L is transformed to

$$L'(\zeta) = -A_x A^{-1} + AL(\zeta)A^{-1}. \tag{20}$$

The gauge transformation rotates the spin chain in an arbitrary direction in the spin space to a spin in the third axis.

In spherical coordinates in the spin space, we obtain

$$S_1 = \cos\theta \qquad S_2 = \sin\theta\cos\varphi \qquad S_3 = \sin\theta\sin\varphi. \tag{21}$$

Then, the gauge transformation is

$$A(x,t) = \mathrm{e}^{-\mathrm{i}\frac{1}{2}\sigma_2\theta}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}\sigma_3\varphi} \qquad A^{-1}(x,t) = \mathrm{e}^{\mathrm{i}\frac{1}{2}\sigma_3\varphi}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}\sigma_2\theta}. \tag{22}$$

Hence,

$$AL(\zeta)A^{-1} = -\mathrm{i}\frac{1}{2}\zeta\sigma_3 - \mathrm{i}\frac{1}{2}\rho^2\zeta^{-1}\big(S_1^2 + S_2^2 - S_3^2\big)\sigma_3 - \mathrm{i}\frac{1}{2}\rho^2\zeta^{-1}\begin{pmatrix} 0 & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & 0 \end{pmatrix} \tag{23}$$

and

$$A_x A^{-1} = -\mathrm{i}\tfrac{1}{2}\sigma_2\theta_x - \mathrm{i}\tfrac{1}{2}\sigma_3\varphi_x\,\mathrm{e}^{\mathrm{i}\sigma_2\theta}. \tag{24}$$

The diagonal elements in (24) do not vanish. From (23) we see that the gauge transformation A transforms $\overrightarrow{S}$ to the third axis. Now introduce another gauge transformation B defined by

$$B = \mathrm{e}^{-\mathrm{i}\frac{1}{2}\sigma_3 f} A \qquad B^{-1} = A^{-1}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}\sigma_3 f}. \tag{25}$$

Here, the diagonal elements of $B_x B^{-1}$ are

$$\big(-\mathrm{i}\tfrac{1}{2}f_x - \mathrm{i}\tfrac{1}{2}\varphi_x\cos\theta\big)\sigma_3 \tag{26}$$

which will vanish by choosing appropriately f. Although the explicit form of f is not known, its existence is evident. Hence, we suppose that the diagonal elements of $B_x B^{-1}$ are 0, and

$$B_x B^{-1} = U = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} \tag{27}$$

which satisfies $U\sigma_3 = -\sigma_3 U$.

Hence, the first operator of the compatibility pair L is transformed into

$$L''(\zeta) = -\mathrm{i}\tfrac{1}{2}\zeta\sigma_3 + U - \mathrm{i}\tfrac{1}{2}\zeta^{-1}w\sigma_3 - \mathrm{i}\zeta^{-1}U_1 \tag{28}$$

where $w = \rho^2\big(S_1^2 + S_2^2 - S_3^2\big)$, $U_1 = \left(\begin{smallmatrix} 0 & u_1 \\ \bar{u}_1 & 0 \end{smallmatrix}\right)$ and $u_1 = 2\mathrm{e}^{-\mathrm{i}f}\sin\theta\cos\theta$.

We will now determine the conserved quantities of (1) through the above transformation. We consider the first compatibility condition

$$v_{1x} = -\mathrm{i}\tfrac{1}{2}(\zeta + \zeta^{-1}w)v_1 + (u - \mathrm{i}\zeta^{-1})v_2 \qquad v_{2x} = \mathrm{i}\tfrac{1}{2}(\zeta + \zeta^{-1}w)v_2 - (\bar{u} + \mathrm{i}\zeta^{-1}\bar{u}_1)v_1 \tag{29}$$

where v is $\psi(x,\zeta)$. Eliminating v_1, we have

$$\begin{aligned} v_{2xx} &- \frac{\bar{u}_x + \mathrm{i}\zeta^{-1}\bar{u}_{1x}}{\bar{u} + \mathrm{i}\zeta^{-1}\bar{u}_1}v_{2x} + \mathrm{i}\frac{1}{2}(\zeta + \zeta^{-1}w)\frac{\bar{u}_x + \mathrm{i}\zeta^{-1}\bar{u}_{1x}}{\bar{u} + \mathrm{i}\zeta^{-1}\bar{u}_1}v_2 \\ &+ \frac{1}{4}(\zeta + \zeta^{-1}w)^2 v_2 + (u - \mathrm{i}\zeta^{-1}u_1)(\bar{u} + \mathrm{i}\zeta^{-1}\bar{u}_1)v_2 = 0. \end{aligned} \tag{30}$$

Set $v_2 = \mathrm{e}^{\mathrm{i}\zeta x + g}$ and substitute the value of v_2 into (30) to get

$$\begin{aligned} g_{xx} + 2\mathrm{i}\zeta g_x + g_x^2 &- \left(\mathrm{i}\frac{1}{2}\zeta + g_x\right)\frac{\bar{u}_x + \mathrm{i}\zeta^{-1}\bar{u}_{1x}}{\bar{u} + \mathrm{i}\zeta^{-1}\bar{u}_1} + \mathrm{i}\frac{1}{2}(\zeta + \zeta^{-1}w)\frac{\bar{u}_x + \mathrm{i}\zeta^{-1}\bar{u}_{1x}}{\bar{u} + \mathrm{i}\zeta^{-1}\bar{u}_1} \\ &+ \frac{1}{2}w + \frac{1}{4}\zeta^{-2}w^2 + (u - \mathrm{i}\zeta^{-1}u_1)(\bar{u} + \mathrm{i}\zeta^{-1}\bar{u}_1) = 0. \end{aligned} \tag{31}$$

As $|\lambda| \to \infty$, g_x has the following asymptotic formula:

$$g_x \equiv \mu = \mu_0 + \mu_1(\mathrm{i}\zeta)^{-1} + \mu_2(\mathrm{i}\zeta)^{-2} + \cdots. \tag{32}$$

Substitute (32) into (31) and compare the coefficient of powers of ζ^{-1} to prove that

$$\mu_0 = 0 \qquad \mu_1 = -|u|^2 - \tfrac{1}{2}w. \tag{33}$$

If we neglect the constant terms, we have

$$\mu_1 = -|u|^2 - \tfrac{1}{2}\rho^2\big(S_1^2 + S_2^2 - S_3^2\big) = -|u|^2 + \rho^2 S_3^2 = -|u|^2 + \tfrac{1}{4}J_3 S_3^2. \tag{34}$$

For the transition coefficient $a(\zeta)$, the following asymptotic expansion as $|\zeta| \to \infty$,

$$\ln a(\zeta) = -\int_{\infty}^{\infty} \mathrm{d}x\mu = -\int_{\infty}^{\infty} \mathrm{d}x(\mu_0 + \mu_1(\mathrm{i}\zeta)^{-1} + \cdots) \tag{35}$$

is known. Since $a(\zeta)$ is independent of t, the coefficients of the asymptotic expansion (35) are conserved quantities I_j, this is $\ln a(\zeta) = I_0 + I_1(\mathrm{i}\zeta)^{-1} + \cdots$. Therefore,

$$I_0 = 0 \qquad I_1 = \int_{\infty}^{\infty}\left(|u|^2 - \frac{1}{4}J_3 S_3^2\right)\mathrm{d}x. \tag{36}$$

Now, we represent $|u|^2$ by the spin chain S_α. $S_\alpha\sigma_\alpha = B^{-1}\sigma_3 B$ follows from (22) and (25). By differentiating with respect to x, we get

$$S_{\alpha x}\sigma_\alpha = -B^{-1}B_x B^{-1}\sigma_3 B + B^{-1}\sigma_3 B_x. \tag{37}$$

Combine (37) with $B_x B^{-1}\sigma_3 = -\sigma_3 B_x B^{-1}$ to get

$$S_{\alpha x}\sigma_\alpha S_\beta\sigma_\beta = -2B^{-1}B_x \qquad S_\beta\sigma_\beta S_{\alpha x}\sigma_\alpha = 2B^{-1}B_x. \tag{38}$$

In addition to the observation that $S_\beta\sigma_\beta S_\alpha\sigma_\alpha = I$ implies

$$S_{\alpha x}\sigma_\alpha S_{\beta x}\sigma_\beta = -4B^{-1}B_x B^{-1}B_x = -4B^{-1}(B_x B^{-1}B_x B^{-1})B = -4B^{-1}U^2B^{-1} = -4U^2. \tag{39}$$

Since U^2 is directly proportional to the identity matrix, we finally obtain

$$4|u|^2 = \overrightarrow{S_x}\cdot\overrightarrow{S_x}. \tag{40}$$

Hence, the first conserved quantity is

$$I_1 = \frac{1}{4}\int_{-\infty}^{+\infty}\big(\overrightarrow{S_x}\cdot\overrightarrow{S_x} - J_3 S_3^2\big)\,\mathrm{d}x. \tag{41}$$

We see from (4) that $2I_1$ is exactly the coordinate representation of the Hamiltonian of the L–L equation with an easy axis.

Next, we consider the spectral representation of $a(\zeta)$. Firstly, we consider the case that $a(\zeta)$ has no zeros. We have proved that $g_x \to 0, a(\zeta) \to 1$ and $\ln a(\zeta) \to 0$ as $|\zeta| \to \infty$. Hence, according to the dispersion relation, we have the asymptotic expansion as $|\zeta| \to \infty$

$$\ln a(\zeta) = -\sum_{j=1}^{\infty}\frac{1}{2\mathrm{i}\pi}\int_{-\infty}^{+\infty}\mathrm{d}\zeta'\ln|a(\zeta')|^2\zeta'^{j-1}\zeta^{-j}. \tag{42}$$

In the same way, the coefficients of ζ^{-j} are conserved quantities. For example

$$I_0 = 0 \qquad I_1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln|a(\zeta')|^2\,\mathrm{d}\zeta'. \tag{43}$$

As for the spectral representation of the Hamiltonian, the same reasoning yields $H = 2I_1$.

Finally, we discuss the case when $a(\zeta)$ has zeros ζ_n. In view of (14) we know that $a(\zeta)$ has two classes of singularities, ζ_n and $-\rho^2\zeta_n^{-1}$, that originate from f_0 and f_3, respectively. In order to satisfy the requirement $a(\zeta) \to 1$, a_d should be of the form

$$a_d(\zeta) = \prod_{n=1}^{N}\frac{(\zeta - \zeta_n)\big(\zeta + \rho^2\zeta_n^{-1}\big)}{(\zeta - \bar{\zeta}_n)\big(\zeta + \rho^2\bar{\zeta}_n^{-1}\big)}. \tag{44}$$

This establishes the asymptotic expansion

$$\ln a_d(\zeta) = \sum_{n=1}^{N}\left[\ln\left(1-\frac{\zeta_n}{\zeta}\right)+\ln\left(1+\frac{\rho^2\zeta_n^{-1}}{\zeta}\right)-\ln\left(1-\frac{\bar{\zeta}_n}{\zeta}\right)-\ln\left(1+\frac{\rho^2\bar{\zeta}_n^{-1}}{\zeta}\right)\right]$$
$$= \sum_{j=1}^{n}\sum_{n=1}^{N}\left[\bar{\zeta}_n^{\,j}+\left(-\rho^2\bar{\zeta}_n^{-1}\right)^j-\left(-\rho^2\zeta_n^{-1}\right)^j-\zeta^{\,j}\right]. \tag{45}$$

Hence, the conserved quantities derived from the discrete spectrum are

$$I_0 = 0 \qquad I_1 = \mathrm{i}\sum\left[\left(\bar{\zeta}_n-\rho^2\bar{\zeta}_n^{-1}\right)-\left(\zeta_n-\rho^2\zeta_n^{-1}\right)\right] = -\mathrm{i}2\sum(\kappa_n-\bar{\kappa}_n). \tag{46}$$

5. The action and angle variables of the discrete spectrum

From the data of inverse scattering, the discrete spectrum is related to the zero of $a(\zeta)$ and the ratio of the Jost function $b_n = \frac{\phi(x,\zeta_n)}{\psi(x,\zeta_n)}$. The former is independent of t and the latter depends on t as the form $b_n(t) = b_n(0)\,\mathrm{e}^{\mathrm{i}4\lambda_n^2 t}$. Introduce an action variable $P_n = F(\kappa_n)$, where F is a function to be determined, and the angle variable $Q_n = \ln b_n = \ln|b_n| + \mathrm{i}\arg b_n$. We obtain the Lie–Poisson brackets which contain the continuous spectral parameter through a similar procedure. The answer is, e.g.

$$\{a(\zeta), b_n\} = -(\kappa\kappa_n+\rho^2)\frac{\zeta}{\lambda_n}\left(\frac{1}{\zeta-\zeta_n}-\frac{1}{\zeta-\eta_n}\right)ab_n \tag{47}$$

where $\eta_n = -\rho^2\zeta_n^{-1}$; nevertheless, $\{b(\zeta), b_n\} = 0$, $\{b_n, b_m\} = 0$.

By (44), the left-hand side of (47) is

$$\{\ln a_c(\zeta), b_n\} + \sum_{m=1}^{N}\left[-\frac{\{\zeta_m, b_n\}}{\zeta-\zeta_m}-\frac{\{\eta_m, b_n\}}{\zeta-\eta_m}+\frac{\{\bar{\zeta}_m, b_n\}}{\zeta-\bar{\zeta}_m}+\frac{\{\bar{\eta}_m, b_n\}}{\zeta-\bar{\eta}_m}\right]. \tag{48}$$

We know that the right-hand side of (47) has two classes of one-order singularities originating from f_0, f_3 respectively. Moreover (48) has simple pole singularities at $\zeta = \zeta_m$, and $\zeta = \eta_m$. Hence,

$$\{\zeta_m, b_n\} = \lambda_m\zeta_m b_n\delta_{mn} \qquad \{\eta_m, b_n\} = -\lambda_m\eta_m b_n\delta_{mn}. \tag{49}$$

We determine P_m through $\{P_m, Q_n\} = -\delta_{mn}$. Since

$$\{F(\kappa_m), \ln b_n\} = \frac{F'(\kappa_m)}{b_n}\{\kappa_m, b_n\} = F'(\kappa_m)\lambda_m^2\delta_{mn} = -\delta_{mn} \tag{50}$$

we have

$$F'(\kappa_m) = -\frac{1}{\lambda_m^2} = -\frac{1}{\kappa_m^2+\rho^2}. \tag{51}$$

Therefore

$$P_m = -\frac{1}{\rho}\arctan\frac{\kappa_m}{\rho}. \tag{52}$$

6. Hamiltonian

From $\partial_t Q_n = \{H, Q_n\}$, we obtain $\partial_t Q_n = 4\lambda_n^2$. Thus, the Hamiltonian which contains the continuous spectral parameter is

$$H(P_m) = \mathrm{i}\rho \tan \rho P_m. \tag{53}$$

From (19), we deduce the Hamiltonian

$$H = -\mathrm{i}4 \sum_m (\kappa_m - \bar{\kappa}_m) + \frac{1}{\pi} \int_{-\infty}^{+\infty} \ln|a(\zeta)|^2 \, \mathrm{d}\zeta. \tag{54}$$

Acknowledgments

The authors thank Professor Qi Minyou and Dr Cai Hao, Wuhan University for valuable advice and discussions. The work is supported by the National Natural Science Foundation of China (nos 10375041 and 10025107).

References

[1] Landau L D and Lifschitz E M 1935 *Phys. Z. Sow.* **8** 153
[2] Faddeev L D 1980 *Mathematical Physics Review Section C: Mathematical Physics Review* (New York: Harwood Academic)
[3] Dodd R K, Eilbeck J C, Gibbon J D and Morris H D 1982 *Solitons and Nonlinear Wave Equation* (New York: Academic)
[4] Takhtajan L A 1977 *Phys. Lett.* A **64** 235
[5] Sklyanin E K 1979 *Sov. Phys.–Dokl.* **24** 107
[6] Sklyanin E K 1979 *Preprint* LOMI E-3-79 (Leningrad)
[7] Chen Z Y, Huang N N and Liu Z Z 1995 *J. Phys.: Condens. Matter* **7** 433
[8] Fogedby H C 1980 *J. Phys. A: Math. Gen.* **13** 1467
[9] Huang N N and Chen Z Y 1990 *J. Phys. A: Math. Gen.* **23** 439
[10] Chen Z D, Chen X J and Huang N N 1999 *J. Math. Phys.* **40** 2933
[11] Shi L N, Cai H, Li C F and Huang N N 2003 *Chin. Phys. Lett.* **20** 1003
[12] Zakharov V E and Takhtajan L A 1979 *Teor. Mat. Fiz.* **38** 26
Zakharov V E and Takhtajan L A 1979 *Theor. Math. Phys.* **38** 17 (Engl. Transl.)